really greater for thick than for thin gold. If the free path is of the order of 10^{-5} cm., that is, comparable with the thickness of the leaf, then the average path in the thicker metal will be longer than in the thinner, and a greater departure would be expected in the thicker metal, as found.

[I am much indebted to my assistant, Mr. J. C. Slater, for his skill in making the readings.]

1 Maxwell, C., Everett, J. D. and Schuster, A., *B. A. Rep.* 1876 (36–63).

2 Bridgman, P. W., *Physic. Rev., Ithaca,* (2), **17,** 1921 (161–194).

AN INTEGRAL EQUALITY AND ITS APPLICATIONS

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The purpose of this note[1] is to deduce an integral equality adjoined to a linear homogeneous differential equation of the second order and to show some applications of such equalities to the question of the distribution of zeros of solutions of such differential equations in the complex plane.

Let $G(z)$ and $K(z)$ be two single-valued and analytic functions of z throughout the region under consideration and take the differential equation

$$\frac{d}{dz}\left[K(z)\frac{dw}{dz}\right] + G(z)w = 0, \tag{1}$$

or the equivalent system

$$\left\{\begin{aligned} \frac{dw_1}{dz} &= \frac{w_2}{K(z)}, \\ \frac{dw_2}{dz} &= -G(z)w_1, \end{aligned}\right. \qquad \text{with} \qquad \left\{\begin{aligned} w_1 &= w, \\ w_2 &= K(z)\frac{dw}{dz}. \end{aligned}\right. \tag{2}$$

From this system we easily deduce the relation

$$\left[\bar{w}_1 w_2\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2 \overline{\left[\frac{dz}{K(z)}\right]} + \int_{z_1}^{z_2} |w_1|^2 G(z)dz = 0, \tag{3}$$

where $\bar{u}$ denotes the conjugate of u. If we put

$$d\mathrm{K} = \frac{dz}{K(z)},\ d\Gamma = G(z)dz, \tag{4}$$

relation (3) becomes

$$\left[\bar{w}_1 w_2\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2\, d\bar{\mathrm{K}} + \int_{z_1}^{z_2} |w_1|^2 d\Gamma = 0, \tag{5}$$

or, split up into real and imaginary parts,

$$\mathfrak{R}\,[\overline{w}_1 w_2]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2\, dK_1 + \int_{z_1}^{z_2} |w_1|^2\, d\Gamma_1 = 0, \tag{6}$$

$$\mathfrak{Im}\,[\overline{w}_1 w_2]_{z_1}^{z_2} + \int_{z_1}^{z_2} |w_2|^2\, dK_2 + \int_{z_1}^{z_2} |w_1|^2\, d\Gamma_2 = 0, \tag{7}$$

where

$$dK = dK_1 + idK_2;\ d\Gamma = d\Gamma_1 + id\,\Gamma_2. \tag{8}$$

We call relation (5) *Green's transform* of the differential equation (1). This relation can be used in many ways for obtaining information concerning the distribution of the zeros of a function $w(z)$, satisfying (1).

Formula (5) enables us to assign regions, below called *zero-free domains* where there can be no zeros of $w(z)$ or $\frac{dw}{dz}$. Some of these ways are indicated below.

The four differential equations

$$dK_1 = 0,\ dK_2 = 0;\ d\Gamma_1 = 0,\ d\Gamma_2 = 0, \tag{9}$$

define four families of curves; the K_1-family and the K_2-family forming the K-net and the Γ_1 and Γ_2-families forming the Γ-net. The two families belonging to the same base net are orthogonal trajectories of each other.

Take a solution $w_a(z)$ of (1) such that

$$W_a(z) = K(z)\, w_a(z) \frac{dw_a}{dz}$$

vanishes at a regular point a in the complex plane. Construct the Riemann surface on which $w_a(z)$ is single-valued and mark the two base nets on the surface. Further, draw all curves on the surface, starting from $z = a$ which do not pass through any of the singular points of the differential equation and which are composed of arcs of curves of the two base nets such that along the whole path one and the same of the following four inequalities is fulfilled, namely

$$1^\circ \left\{\begin{matrix} dK_1 \geq 0, \\ d\Gamma_1 \leq 0; \end{matrix}\right| \quad 2^\circ \left\{\begin{matrix} dK_1 \leq 0, \\ d\Gamma_1 \geq 0; \end{matrix}\right| \quad 3^\circ \left\{\begin{matrix} dK_2 \geq 0, \\ d\Gamma_2 \geq 0; \end{matrix}\right| \quad 4^\circ \left\{\begin{matrix} dK_2 \leq 0, \\ d\Gamma_2 \leq 0. \end{matrix}\right. \tag{10}$$

We agree to smooth the corners of the path-curves, when necessary, in order to preserve the continuity of the tangent along the curve. Such curves we call *standard paths* and designate the four different kinds (distinguished by their characteristic inequalities) respectively

$$SK_1^+,\ SK_1^-,\ SK_2^+ \text{ and } SK_2^-. \tag{11}$$

The points on the surface which belong to at least one standard path, emanating from $z = a$, together form the *standard domain* $D(a)$ of a. A

point on the boundary of $D(a)$ is counted part of the standard domain provided it is not a singular point of the differential equation and is different from a itself. In view of (6) and (7) we obtain the result:

There is no zero of $W_a(z)$ in the standard domain of a.

Here is another application of the Green's transform. Suppose $G(z)$ and $K(z)$ to be analytic and, furthermore, real on an interval (a,b) of the real axis. Then there are solutions of (1) which are real on the same interval. Draw all standard paths of the third and the fourth kind, SK_2^+ and SK_2^-, which emanate from the points on (a,b). The points on these paths, not including possible singular points and the points on (a,b), form a zero-free region for all solutions real on (a,b) in virtue of formula (7). For this statement we have only used the fact that for real solutions $\mathfrak{Im}\,[\overline{w_1(z)}\; w_2\,(z)] = 0$ on (a,b). The generalizations are obvious.

The Standard Domain Is Covariant under Conformal Transformation.—By this we understand that a change of independent variable from z to Z by putting $Z = F\,(z)$ which preserves the form of the differential system (2), carries the standard domain of a point z_0 in the z-plane over into the standard domain of the corresponding point Z_0 in the Z-plane and these two domains are the conformal maps of each other by the transformation $Z = F\,(z)$ and its inverse, provided $F\,(z)$ is regular in $D\,(z_0)$ and $F'\,(z) \neq 0$ there. This is obvious because Green's transform is an invariant under such a transformation and the base nets in the two planes correspond to each other by the conformal transformation.

The standard domain is not covariant under a simultaneous change of dependent and independent variable. By a fortunate choice of such variables it is often possible to determine zero-free domains of such an extent that there is comparatively little freedom left for placing possible zeros; thus one is able to obtain a fairly good qualitative description of the arrangement of the zeros in the plane. The transformation

$$(12)\quad \begin{cases} dZ = \sqrt{\dfrac{G(z)}{K(z)}}\, dz, \\ W = \sqrt{S(Z)}\; w, \\ S(Z) = \sqrt{G(z)K(z)}, \end{cases}$$

often yields good service for investigation of the distribution of zeros of solutions in the neighborhood of an irregular singular point of the differential equation.

[1] This note is an abstract of a paper offered to *Trans. Amer. Math. Soc.* for publication.